Projection on the kernel of a matrix J

$$J \in \mathbf{R}^{m\times n}$$

SVD decomposition

$$\begin{aligned}
J &= USV^T \quad U \in O(m) \quad V \in O(n) \\
J^+ &= VS^+U^T \\
J^+J &= VS^+SV^T \\
&= V \begin{pmatrix} I_m & O_{m\times n-m} \\ O_{n-m\times m} & O_{n-m\times n-m} \end{pmatrix} V^T
\end{aligned}$$

where m is the (full) rank of J.

$$\begin{aligned}
I_n - J^+J &= V \begin{pmatrix} O_m & O_{m\times n-m} \\ O_{n-m\times m} & I_{n-m} \end{pmatrix} V^T \\
&= \begin{pmatrix} V_1 & V_0 \end{pmatrix} \begin{pmatrix} O_m & O_{m\times n-m} \\ O_{n-m\times m} & I_{n-m} \end{pmatrix} \begin{pmatrix} V_1^T \\ V_0^T \end{pmatrix} \\
&= V_0V_0^T
\end{aligned}$$

Projection on the kernel of a matrix J

$$J \in \mathbf{R}^{m\times n}$$

$$J^TP = QR$$

where $P \in O(m)$ is a permutation matrix, $Q \in O(n)$ and $R \in \mathbf{R}^{n\times m}$ is an upper triangular matrix.

$$J = PLQ^T$$

where $L = R^T$.

$$\begin{aligned}
L &= \begin{pmatrix} L_1 & 0 \end{pmatrix} \\
L^+ &= \begin{pmatrix} L_1^-1 \\ 0 \end{pmatrix} \\
L^+L &= \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

where r is the rank of L.

$$\begin{aligned}
J^+J &= \begin{pmatrix} Q_1 & Q_0 \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Q_1 & Q_0 \end{pmatrix}^T \\
&= Q_1Q_1^T \\
I_n - J^+J &= Q_0Q_0^T
\end{aligned}$$