# Continuous Collision Checking

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## 1 Definitions and Notation

Given a robot as a tree of joints moving in a workspace, and given a path for this robot between two configurations, we wish to establish whether the path is collision free with the environment or for self-collision.

For each pair body a - body b, we will validate intervals by

1. computing the distance between bodies at a given parameter, and
2. bound from above the velocity of all points of body a in the frame of body b.

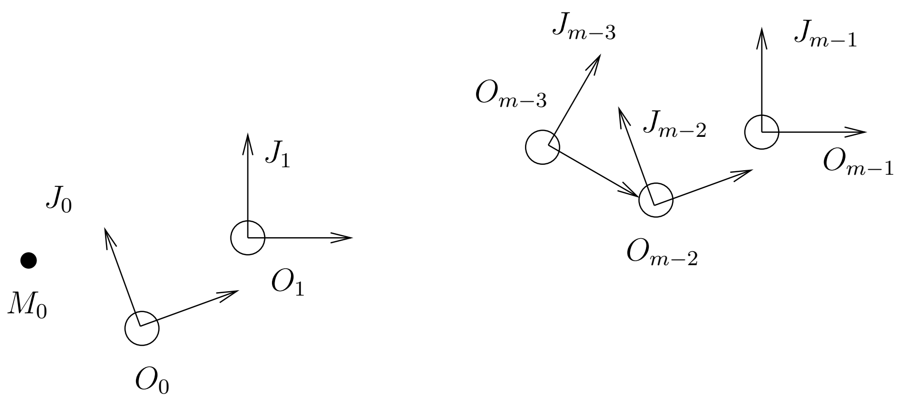
Let us denote by

- $J_a$ and $J_b$ the two joints holding
- bodies $\mathcal{B}_a$ and $\mathcal{B}_b$ of the pair to check for collision,
- $J_0 = J_a, J_1, \cdots, J_{m-1} = J_b$, the list of joints linking $J_a$ to $J_b$,
- $\mathcal{C}$ the configuration space of the robot,
- $P : [0, T] \to \mathcal{C}$, the path to check for collision,
- $\mathbf{q}_i = P(0)$ and $\mathbf{q}_g = P(T)$ the end configurations of the path to check.

## 2 Constant velocity

In this section, we assume that along the path $P$, each joint $J_i$ rotates or translates at constant linear and/or angular velocity in the reference frame of its neighbor. We thus denote for $i = 1, \cdots, m-1$,

- $\mathbf{v}_{i-1/i}$, the constant linear velocity, and
- $\omega_{i-1/i}$, $i = 1, \cdots, m-1$ the constant angular velocity of joint $J_{i-1}$ in the reference frame of joint $J_i$.




# 3 Upper bound on relative velocity

Let $P_0$ be a point fixed in reference frame $J_0$ of coordinates $p_0$ in the local frame of $J_0$. The coordinate of $P_0$ in the frame of $J_{m-1}$ is given by

$$P_{0/m-1} = M_{m-2/m-1} M_{m-3/m-2} \cdots M_{1/2} M_{0/1} \begin{pmatrix} m_0 \\ 1 \end{pmatrix} \tag{1}$$

where

- $M_{i/i+1} = \begin{pmatrix} R_{i/i+1} & T_{i/i+1} \\ 0\ 0\ 0 & 1 \end{pmatrix}$ is the homogeneous matrix representing the position of Joint $J_i$ in the reference frame of $J_{i+1}$,
- $M_{i/i+1} \in SO(3)$ is a rotation matrix, and
- $T_{i/i+1} \in \mathbb{R}^3$ is a translation vector.

Differentiating (1), we get

$$\begin{aligned}
\begin{pmatrix} \dot{P}_{0/m-1} \\ 0 \end{pmatrix} &= \begin{pmatrix} [\omega_{m-2/m-1}]_\times R_{m-2/m-1} & \mathbf{v}_{m-2/m-1} \\ 0\ 0\ 0 & 0 \end{pmatrix} \cdots M_{1/2} M_{0/1} \begin{pmatrix} m_0 \\ 1 \end{pmatrix} && (2) \\
&+ M_{m-2/m-1} \begin{pmatrix} [\omega_{m-3/m-2}]_\times R_{m-3/m-2} & \mathbf{v}_{m-3/m-2} \\ 0\ 0\ 0 & 0 \end{pmatrix} \cdots M_{0/1} \begin{pmatrix} m_0 \\ 1 \end{pmatrix} && (3) \\
&+ \cdots && (4) \\
&+ M_{m-2/m-1} \cdots M_{1/2} \begin{pmatrix} [\omega_{0/1}]_\times R_{0/1} & \mathbf{v}_{0/1} \\ 0\ 0\ 0 & 0 \end{pmatrix} \begin{pmatrix} m_0 \\ 1 \end{pmatrix} && (5)
\end{aligned}$$

where

- $[\omega_{i/i+1}]_\times$ is the antisymmetric matrix corresponding to the cross product by vector $\omega_{i/i+1} \in \mathbb{R}^3$, representing the angular velocity of $J_i$ with respect to $J_{i+1}$,
- $\mathbf{v}_{i/i+1} = \dot{T}_{i/i+1}$ is the linear velocity of the origin of $J_i$ in the reference frame of $J_{i+1}$.

### 3.1 A few properties of rigid-body transformations

Let $M_1 = \begin{pmatrix} R_1 & T_1 \\ 0\ 0\ 0 & 1 \end{pmatrix}$, $M_2 = \begin{pmatrix} R_2 & T_2 \\ 0\ 0\ 0 & 1 \end{pmatrix}$, and $M_3 = \begin{pmatrix} R_3 & T_3 \\ 0\ 0\ 0 & 1 \end{pmatrix}$ be three homogeneous matrices such that

$$M_3 = M_1 M_2 = \begin{pmatrix} R_1\ R_2 & R_1\ T_2 + T_1 \\ 0\ 0\ 0 & 1 \end{pmatrix}$$

We notice that

$$\|T_3\| \leq \|T_1\| + \|T_2\| \tag{6}$$

Let $m \in \mathbb{R}^3$, and $p \in \mathbb{R}^3$ such that

$$\begin{pmatrix} p \\ 1 \end{pmatrix} = M_1 \begin{pmatrix} m \\ 1 \end{pmatrix}$$

Then

$$\|p\| \leq \|T_1\| + \|m\| \tag{7}$$

### 3.2 Upper-bound computation

From properties (6-7) and expression (2-5), we get

$$\begin{aligned}
\|\dot{P}_{0/m-1}\| \quad \leq \quad & \|\mathbf{v}_{0/1}\| + \|\omega_{0/1}\|\ \|m_0\| \\
& +\|\mathbf{v}_{1/2}\| + \|\omega_{1/2}\| \left(\|m_0\| + \|T_{0/1}\|\right) \\
& +\|\mathbf{v}_{2/3}\| + \|\omega_{2/3}\| \left(\|m_0\| + \|T_{0/1}\| + \|T_{1/2}\|\right) \\
& +\cdots \\
& +\|\mathbf{v}_{m-2/m-1}\| + \|\omega_{m-2/m-1}\|\ \left(\|m_0\| + \|T_{1/2}\| + \cdots + \|T_{m-2/m-1}\|\right)
\end{aligned}$$

Notice that red variables correspond to joint variable derivatives and depend on the path, while black expressions are constant for a given kinematic chain.

If we define the radius of body $\mathcal{B}_a$ as the maximum distance of all points of the body to the center of the joint:

$$r_0 = \sup\left\{\|m_0\|, m_0 \in \mathcal{B}_a\right\},$$

we get

$$\begin{aligned}
\|\dot{P}_{0/m-1}\| \quad \leq \quad & \|\mathbf{v}_{0/1}\| + \|\omega_{0/1}\|\ r_0 \\
& +\|\mathbf{v}_{1/2}\| + \|\omega_{1/2}\| \left(r_0 + \|T_{0/1}\|\right) \\
& +\|\mathbf{v}_{2/3}\| + \|\omega_{2/3}\| \left(r_0 + \|T_{0/1}\| + \|T_{1/2}\|\right) \\
& +\cdots \\
& +\|\mathbf{v}_{m-2/m-1}\| + \|\omega_{m-2/m-1}\|\ \left(r_0 + \|T_{1/2}\| + \cdots + \|T_{m-2/m-1}\|\right)
\end{aligned}$$

# 4 If $J_a$ is not ancestor nor descendant of $J_b$

If the robot is a tree of joints, $J_a$ and $J_b$ may lie on different branches and therefore not be ancestor nor descendant of one another. In this case, in the sequence, $J_0$,..., $J_{m-1}$, any joint can be the child or the parent of its predecessor in the list. Let $J_i$ and $J_{i+1}$ be two consecutive joints in $J_0$,..., $J_{m-1}$. Notice that

$$\|T_{i/i+1}\| = \|T_{i+1/i}\|$$

Without loss of generality, we can then assume that $J_{i+1}$ is the child of $J_i$. We define

$$\begin{aligned} M &= J_{i+1}\texttt{->positionInParentFrame ()} \\ T &= M[0:3,3] \end{aligned}$$

$T$ is the coordinate of the origin of $J_{i+1}$ expressed in frame $J_i$.

- if $J_{i+1}$ is a rotation or SO(3) joint,

$$\|T_{i/i+1}\| = \|T\|,$$

- if $J_{i+1}$ is a translation joint bounded in interval $[v_{min}, v_{max}]$, $\|T_{i/i+1}\|$ is the maximum of two values computed as follows: let

$$\mathbf{u} = M[0:3,0]$$

  $\mathbf{u}$ is the direction of translation of $J_{i+1}$ expressed in frame $J_i$. Then

$$\|T_{i/i+1}\| \leq \max(\|T + v_{min}\mathbf{u}\|, \|T + v_{max}\mathbf{u}\|)$$